Dual Dig Level I (2009) - Solutions

Jnless otherwise stated or implied, assume that all graphs are in the usual *xy*-plane.

1. The quotient of 2 numbers is equivalent to $\frac{7}{4}$. If the numerator and the denominator are each increased by 20, the quotient becomes equal to $\frac{11}{8}$. Find the original numbers.

Solution: $\frac{x}{y}=\frac{7}{4}\Rightarrow\frac{x+20}{y+20}=\frac{11}{8}\Rightarrow 8x-11y=60$. But, from the original equation, $x=\frac{7y}{4}$, so by substitution, $8\left(\frac{7y}{4}\right)-11y=60\Rightarrow y=20$, and eventually, $x=35$. Thus, the original quotient was $\frac{35}{20}$.

2. If $\frac{A}{B}+\frac{4}{3}+\frac{9}{2}=\left(\frac{A}{B}\right)\left(\frac{4}{3}\right)\left(\frac{9}{2}\right)$, find $\frac{A}{B}$ in lowest terms.

Solution: This simplifies to: $\frac{A}{B}+\frac{35}{6}=6\frac{A}{B}\Rightarrow 5\frac{A}{B}=\frac{35}{6}\Rightarrow\frac{A}{B}=\frac{7}{6}$

3. A line with slope 2 intersects a line with slope 6 at the point $(40,30)$. What is the distance between the *x*-intercepts of the 2 lines?

Solution:

Equation of 1st line: $y=2x-50$, so its *x*-intercept is $(25,0)$.

Equation of 2nd line: $y=6x-210$, so its *x*-intercept is $(35,0)$.

Since 35 – 25 = 10, the distance is 10 units.

4. Solve the system: $\begin{cases} x-2y+3z=9 \\ 3y-x=-4 \\ 2x-5y+5z=17 \end{cases}$

Solution: You could use matrices to solve this system, starting with the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Answer: $x=1$, $y=-1$, and $z=2$.

5. A positive real decimal number has the property that, when its decimal point is moved four places to the right, the result is four times the reciprocal of the original number. Find the original decimal number.

Solution: Let x be the original decimal number. Then, $10{,}000x$ is the number that is obtained when the decimal point is moved four places to the right.

$$10{,}000x = 4\left(\frac{1}{x}\right)$$
$$10{,}000x^2 = 4$$
$$x^2 = \frac{1}{2500}$$
$$x = \frac{1}{50} \quad (\text{Remember, } x > 0.)$$
$$x = 0.02$$

Answer: 0.02.

6. A bathroom scale is set too high. When Tweedledum stands on the scale, it reads 180 pounds. When Tweedledee (alone) stands on the scale, it reads 240 pounds. When they both stand on the scale, the scale reads 400 pounds. How many pounds too high is the scale set?

Solution 1: Imagine Tweedledum getting on the scale and then Tweedledee joining him. The 180 pounds that the scale reads for Tweedledum includes the "excess" by which the scale is set too high. After Tweedledee joins him, the scale reads 400 pounds; this means that Tweedledee actually weighs 220 pounds $(400-180=220)$. Since the scale reads 240 pounds when Tweedledee is on it alone, that means that the scale is set 20 pounds too high $(240-220=20)$. Answer: 20 pounds.

Solution 2:

Let x = Tweedledum's true weight (in pounds).
Let y = Tweedledee's true weight (in pounds).
Let z = the "excess" by which the scale is set too high (in pounds).

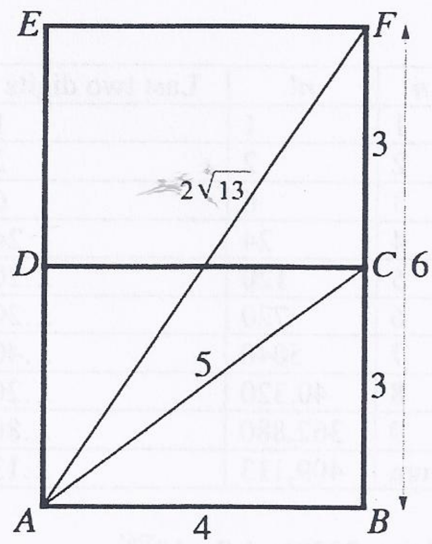
Solve the following system for z:

$$\begin{cases} x+z=180 \\ y+z=240 \\ x+y+z=400 \end{cases}$$

The second and third equations give us $x=160$.
Then, the first equation gives us $z = 20$ pounds.
Answer: 20 pounds.

7. $ABCD$ and $DCFE$ are coplanar rectangles with $AB = 4$, $AC = 5$, and $BC = CF$. (Point B and Point F are distinct points.) What is the exact length of $\overline{AF}$ in simplest form?

Solution: In rectangle $ABCD$, the Pythagorean Theorem gives $BC = 3$. Thus, the entire rectangle $ABFE$ has dimensions 4 by 6 with $\overline{AF}$ as a diagonal. Again, using the Pythagorean Theorem, $AF = \sqrt{52} = 2\sqrt{13}$.



8. Find the values of three nonzero distinct digits, "A," "B," and "C," so that the square of the number formed by BC (that is: B in the tens place and C in the ones place) is equal to the number ABC.

Solution: A little trial and error shows that 32 is the lowest positive integer whose square consists of more than three digits: $(32)^2 > 999$. Thus, BC must be smaller than 32. Also, "C" can only be 0 (which we eliminate), 1, 5, or 6, since these are the only digits that, when squared, end in themselves (awkward, but true!) Finally, guess and check shows that only 25 satisfies our requirements: $(25)^2 = 625$. Thus, A = 6, B = 2, and C = 5.

9. For a certain two digit number, the tens digit is twice the units digit. If 36 is subtracted from the original number, the digits will be interchanged. Find the original number.

Solution: Let xy = the original number, and yx = the 'interchanged' number; the adjacency of x and y does not indicate multiplication here. Then, $10x + y - 36 = 10y + x \Rightarrow x - y = 4$. But, $x = 2y$, $\Rightarrow$ $2y - y = 4 \Rightarrow y = 4$. Since $x = 2y$, $\Rightarrow x = 8$. Thus, the original number is 84.

10. If n is a positive integer, then $n!$, called "n factorial," is the product of the integers from 1 through n. Give the digit in the tens place of the integer whose value is $1!+2!+3!+4!+...+2009!$, which is the sum of the factorials of the first 2009 positive integers.

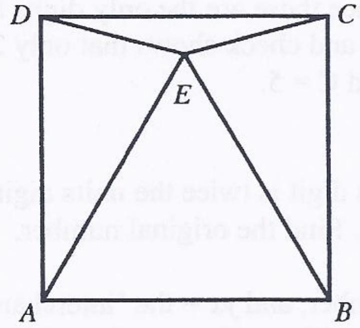
Solution: Observe that $n!$ ends in at least two '0's whenever $n \geq 10$, since those factorials are obtained by multiplying 2, 5, and 10 (as well as other positive integers) and thus must be multiples of 100. Therefore, the digit we want is the digit in the tens place of $1!+2!+3!+4!+...+9!$. In fact, it is sufficient to compute the last two digits of the factorials as you successively compute them by multiplying in 2, 3, etc.

n	$n!$	Last two digits
1	1	1
2	2	2
3	6	6
4	24	24
5	120	...20
6	720	...20
7	5040	...40
8	40,320	...20
9	362,880	...80
Sum	409,113	...13

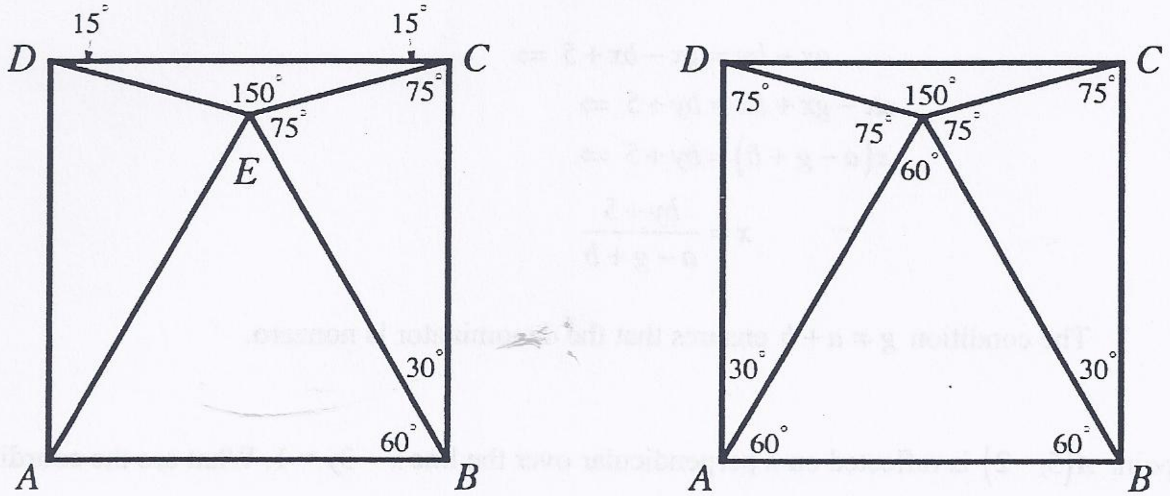
Note: 2009! has 5766 digits. $2009! \approx 1.7 \times 10^{5765}$.

Answer: 1.

11. Point E is a point inside the square $ABCD$ such that triangle EAB is an equilateral triangle. Find the measure of angle CED.



Solution: Observe that triangles *AED, BEC,* and *ECD* are isosceles triangles. Answer: 150° , or $\dfrac{5\pi}{6}$.



12. Find the product of: $\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\left(1-\dfrac{1}{4^2}\right)\ldots\ldots\left(1-\dfrac{1}{10^2}\right)$, and simplify.

Solution: Simplify each quantity, then very carefully cross-cancel.

$$\frac{3}{4}\bullet\frac{8}{9}\bullet\frac{15}{16}\bullet\frac{24}{25}\bullet\frac{35}{36}\bullet\frac{48}{49}\bullet\frac{63}{64}\bullet\frac{80}{81}\bullet\frac{99}{100}=\frac{11}{20}$$

13. A new mathematical operation is formed and defined as: $a\otimes b=(2a+b)^2$.

Calculate: $\sqrt{(3\otimes 2)\otimes(2\otimes 3)}$.

Solution: $3\otimes 2=\left(2(3)+2\right)^2=64$, and $2\otimes 3=\left(2(2)+3\right)^2=49$, so

$\sqrt{\left(64\otimes 49\right)^2}=2(64)+49=177$.

14. Assuming that all variables represent positive real numbers, and $g \neq a+b$, solve for x: $ax - by = gx - bx + 5$.

Solution: Rearrange and factor:

$$\begin{aligned} ax - by = gx - bx + 5 &\Rightarrow \\ ax - gx + bx = by + 5 &\Rightarrow \\ x(a - g + b) = by + 5 &\Rightarrow \\ x = \frac{by+5}{a-g+b} & \end{aligned}$$

The condition $g \neq a+b$ ensures that the denominator is nonzero.

15. The point $A(5,-2)$ is reflected on a perpendicular over the line $x - 3y = 1$. What are the coordinates of the reflected point A'?

Solution: The slope of the original line is $\frac{1}{3}$, thus the slope of the perpendicular line is -3, or $\frac{3}{-1}$. Utilizing the slope ratio as $\frac{rise}{run}$, and starting from the point $A(5,-2)$, the next integer-value point on the perpendicular line is (4, 1). (IMPORTANT NOTE: (4, 1) happens to also satisfy the first equation, thus it becomes the midpoint between point A and its reflection.) Utilizing the slope ratio $\frac{3}{-1}$ one more time, but starting from the point (4, 1), we arrive at the point (3, 4). This means that (3, 4) is the reflection of point $A(5,-2)$ over the line $x - 3y = 1$. Answer: (3, 4).

16. The graph of $y = ax^3 + bx^2 + cx + d$ includes the points $(-1,0)$, $(1,0)$, and $(0,3)$. Assume that a, b, c, and d represent real constants. What must be the value of b?

Solution:

$(-1,0)$ is on the graph $\Rightarrow 0 = a(-1)^3 + b(-1)^2 + c(-1) + d \Rightarrow 0 = -a + b - c + d$.

$(1,0)$ is on the graph $\Rightarrow 0 = a(1)^3 + b(1)^2 + c(1) + d \Rightarrow 0 = a + b + c + d$.

Add equals to equals to get equals:

$$\begin{cases} 0 = -a + b - c + d \\ 0 = a + b + c + d \end{cases}$$

$$0 = \quad 2b \quad + 2d \Rightarrow b + d = 0.$$

$(0,3)$ is on the graph $\Rightarrow 3 = a(0)^3 + b(0)^2 + c(0) + d \Rightarrow d = 3$. Therefore, $b = -3$.

Answer: -3.

17. An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is ten square meters, what is the area of the hexagon? Hint: Draw the triangle with line segments connecting the midpoints of the sides.

Solution:

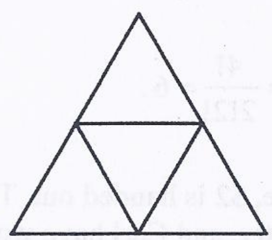
Let s = the side length of the equilateral triangle. When we connect the midpoints of the sides of the triangle, we decompose the triangle into four congruent equilateral triangles of side length $\frac{s}{2}$. (See the figure on the left.)

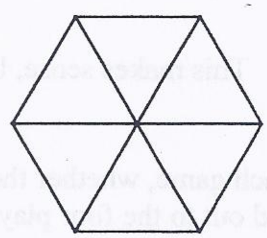
The original triangle has perimeter $3s$. This must also be the perimeter of the hexagon. Because the regular hexagon consists of six congruent sides, it must have side length $\frac{3s}{6}$, or $\frac{s}{2}$. (Now, see the figure on the right.) The hexagon can be decomposed into six congruent equilateral triangles that are congruent to the four congruent triangles seen in the figure on the left.

The ratio between the areas of the hexagon and the original triangle is $\frac{6}{4}$, or $\frac{3}{2}$.

The area of the original triangle is ten square meters, so the area of the hexagon is:

$\left(\frac{3}{2}\right)(10) = 15$ square meters.





18. Which of the following three numbers is greatest in value: $\sqrt{2}$, $\sqrt[3]{3}$, or $\sqrt[6]{6}$?

Solution: Observe: If a and b are positive real numbers, then $a > b \Leftrightarrow a^6 > b^6$.

It is sufficient to compare the sixth powers of the given numbers:

$$\left(\sqrt{2}\right)^6 = \left(2^{1/2}\right)^6 = 2^3 = 8$$

$$\left(\sqrt[3]{3}\right)^6 = \left(3^{1/3}\right)^6 = 3^2 = 9 \quad \Rightarrow \quad \text{Answer: } \sqrt[3]{3}.$$

$$\left(\sqrt[6]{6}\right)^6 = 6$$

Note 1: $\sqrt{2} \approx 1.41$, $\sqrt[3]{3} \approx 1.44$, and $\sqrt[6]{6} \approx 1.35$.

Note 2: Calculus can be used to show that $\sqrt[3]{3}$, $\sqrt[4]{4}$, $\sqrt[5]{5}$, ... make up a decreasing sequence that approaches 1. Experiment with this on a calculator.

19. Solve for *x*: $2x+3 \le |x+5| \le 6x-9$. Consider only real values for *x*.

Solution: The problem splits into two different inequalities: $|x+5| \ge 2x+3$ AND $|x+5| \le 6x-9$.

$|x+5| \ge 2x+3 \Rightarrow x+5 \ge 2x+3$ or $x+5 \le -(2x+3) \Rightarrow x \le 2$, while

$|x+5| \le 6x-9 \Rightarrow -(6x-9) \le x+5 \le 6x-9$ (We may assume $6x-9 \ge 0$.) $\Rightarrow x \ge \frac{14}{5}$.

Since the original problem was a conjunction, we are looking for the intersection between the sets of real numbers corresponding to $x \le 2$ AND $x \ge \frac{14}{5}$ … and the intersection set is empty! There are no real solutions, and the solution set is the empty or null set, $\varnothing$.

20. Al, Betty, Carl, and Dina play a series of one-on-one basketball games, and their P.E. teacher hands out money based on the results. Each player plays a game with each other player exactly once. For each game, if there is a winner, then the winner wins \$2 and the loser gets \$0; in the event of a tie, each player gets \$1. After all the games have been played, Al has won \$3, Betty has won \$4, and Carl has won \$1. How much money has Dina won?

Solution:

Let *A*, *B*, *C*, and *D* represent Al, Betty, Carl, and Dina, respectively.
A total of six games have been played:

A vs. *B*		
A vs. *C*	*B* vs. *C*	
A vs. *D*	*B* vs. *D*	*C* vs. *D*

This makes sense, because $\binom{4}{2}$ or ${}_4C_2 = \frac{4!}{2!2!} = 6$.

For each game, whether there is a winner or a tie, \$2 is handed out. Therefore, a total of \$12 is handed out to the four players. Together, Al, Betty, and Carl have won \$8, so Dina had to win the remaining \$4.

Challenge: There are two possible ways for the six games to have turned out collectively. What are they? Hint: Grids may help.

Answer: \$4.